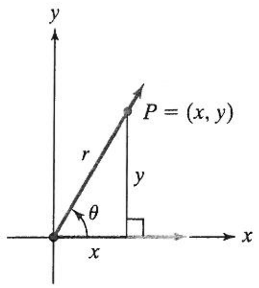
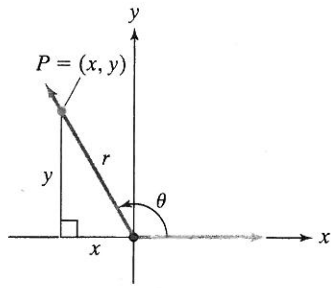
104. From the top of a 250-foot lighthouse, a plane is sighted overhead and a ship is observed directly below the plane. The angle of elevation of the plane is 22° and the angle of depression of the ship is 35°. Find **a.** the distance of the ship from the lighthouse; **b.** the plane's height above the water. Round to the nearest foot.

Preview Exercises

Exercises 105–107 will help you prepare for the material covered in the next section. Use these figures to solve Exercises 105–106.



(a) θ lies in quadrant I.



(b) θ lies in quadrant II.

105. a. Write a ratio that expresses $\sin\theta$ for the right triangle in **Figure (a)**.

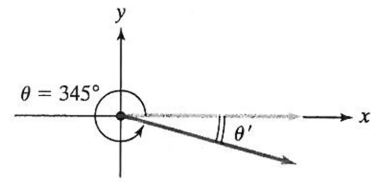
b. Determine the ratio that you wrote in part (a) for **Figure (b)** with $x = -3$ and $y = 4$. Is this ratio positive or negative?

106. a. Write a ratio that expresses $\cos\theta$ for the right triangle in **Figure (a)**.

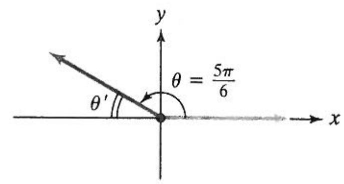
b. Determine the ratio that you wrote in part (a) for **Figure (b)** with $x = -3$ and $y = 5$. Is this ratio positive or negative?

107. Find the positive angle θ' formed by the terminal side of θ and the x-axis.

a.



b.



Section 5.3 Trigonometric Functions of Any Angle

Objectives

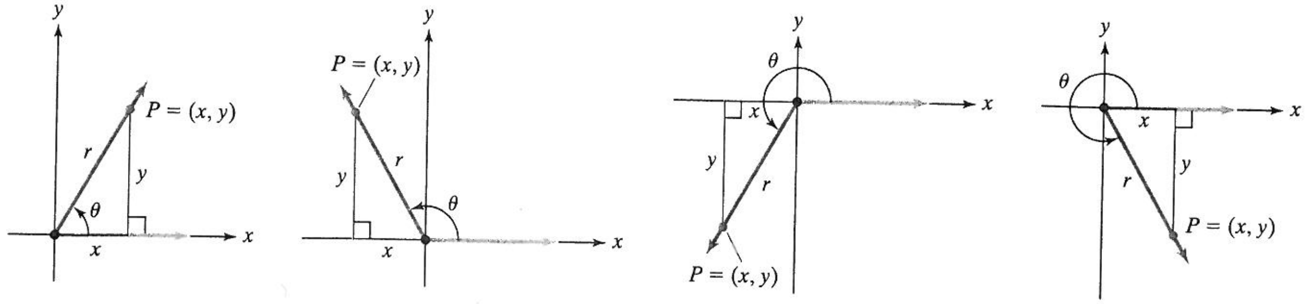
1. Use the definitions of trigonometric functions of any angle.
2. Use the signs of the trigonometric functions.
3. Find reference angles.
4. Use reference angles to evaluate trigonometric functions.

There is something comforting in the repetition of some of nature's patterns. The ocean level at a beach varies between high and low tide approximately every 12 hours. The number of hours of daylight oscillates from a maximum on the summer solstice, June 21, to a minimum on the winter solstice, December 21. Then it increases to the same maximum the following June 21. Some believe that cycles, called biorhythms, represent physical, emotional, and intellectual aspects of our lives. Throughout the remainder of this chapter, we will see how the trigonometric functions are used to model phenomena that occur again and again. To do this, we need to move beyond right triangles.

1 Use the definitions of trigonometric functions of any angle.

Trigonometric Functions of Any Angle

In the last section, we evaluated trigonometric functions of acute angles, such as that shown in **Figure 5.32(a)**. Note that this angle is in standard position. The point $P = (x, y)$ is a point r units from the origin on the terminal side of θ. A right triangle is formed by drawing a line segment from $P = (x, y)$ perpendicular to the x-axis. Note that y is the length of the side opposite θ and x is the length of the side adjacent to θ.



(a) θ lies in quadrant I. **(b)** θ lies in quadrant II. **(c)** θ lies in quadrant III. **(d)** θ lies in quadrant IV.

Figure 5.32

Figures 5.32(b), **(c)**, and **(d)** show angles in standard position, but they are not acute. We can extend our definitions of the six trigonometric functions to include such angles, as well as quadrantal angles. (Recall that a quadrantal angle has its terminal side on the x-axis or y-axis; such angles are *not* shown in **Figure 5.32**.) The point $P = (x, y)$ may be any point on the terminal side of the angle θ other than the origin, $(0, 0)$.

Study Tip

If θ is acute, we have the right triangle shown in **Figure 5.32(a)**. In this situation, the definitions in the box are the right triangle definitions of the trigonometric functions. This should make it easier for you to remember the six definitions.

Definitions of Trigonometric Functions of Any Angle

Let θ be any angle in standard position and let $P = (x, y)$ be a point on the terminal side of θ. If $r = \sqrt{x^2 + y^2}$ is the distance from $(0, 0)$ to (x, y), as shown in **Figure 5.32**, the **six trigonometric functions of θ** are defined by the following ratios:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}, \ y \neq 0$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}, \ x \neq 0$$

$$\tan\theta = \frac{y}{x}, \ x \neq 0 \qquad \cot\theta = \frac{x}{y}, \ y \neq 0.$$

The ratios in the second column are the reciprocals of the corresponding ratios in the first column.

Because the point $P = (x, y)$ is any point on the terminal side of θ other than the origin, $(0, 0)$, $r = \sqrt{x^2 + y^2}$ cannot be zero. Examine the six trigonometric functions defined above. Note that the denominator of the sine and cosine functions is r. Because $r \neq 0$, the sine and cosine functions are defined for any angle θ. This is not true for the other four trigonometric functions. Note that the denominator of the tangent and secant functions is x: $\tan\theta = \frac{y}{x}$ and $\sec\theta = \frac{r}{x}$. These functions are not defined if $x = 0$. If the point $P = (x, y)$ is on the y-axis, then $x = 0$. Thus, the tangent and secant functions are undefined for all quadrantal angles with terminal sides on the positive or negative y-axis. Likewise, if $P = (x, y)$ is on the x-axis, then $y = 0$, and the cotangent and cosecant functions are undefined: $\cot\theta = \frac{x}{y}$ and $\csc\theta = \frac{r}{y}$. The cotangent and cosecant functions are undefined for all quadrantal angles with terminal sides on the positive or negative x-axis.

EXAMPLE 1 Evaluating Trigonometric Functions

Let $P = (-3, -5)$ be a point on the terminal side of θ. Find each of the six trigonometric functions of θ.

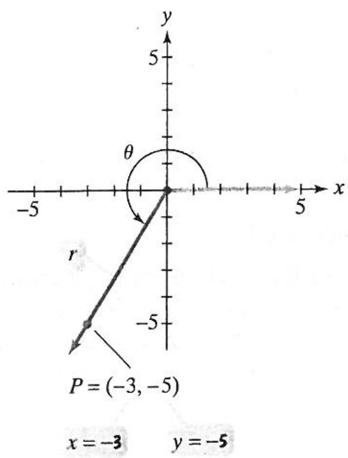


Figure 5.33

Solution The situation is shown in **Figure 5.33**. We need values for x, y, and r to evaluate all six trigonometric functions. We are given the values of x and y. Because $P = (-3, -5)$ is a point on the terminal side of θ, $x = -3$ and $y = -5$. Furthermore,

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}.$$

Now that we know x, y, and r, we can find the six trigonometric functions of θ. Where appropriate, we will rationalize denominators.

$$\sin\theta = \frac{y}{r} = \frac{-5}{\sqrt{34}} = -\frac{5}{\sqrt{34}}\cdot\frac{\sqrt{34}}{\sqrt{34}} = -\frac{5\sqrt{34}}{34} \qquad \csc\theta = \frac{r}{y} = \frac{\sqrt{34}}{-5} = -\frac{\sqrt{34}}{5}$$

$$\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{34}} = -\frac{3}{\sqrt{34}}\cdot\frac{\sqrt{34}}{\sqrt{34}} = -\frac{3\sqrt{34}}{34} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{34}}{-3} = -\frac{\sqrt{34}}{3}$$

$$\tan\theta = \frac{y}{x} = \frac{-5}{-3} = \frac{5}{3} \qquad \cot\theta = \frac{x}{y} = \frac{-3}{-5} = \frac{3}{5}$$

Check Point 1 Let $P = (1, -3)$ be a point on the terminal side of θ. Find each of the six trigonometric functions of θ.

How do we find the values of the trigonometric functions for a quadrantal angle? First, draw the angle in standard position. Second, choose a point P on the angle's terminal side. The trigonometric function values of θ depend only on the size of θ and not on the distance of point P from the origin. Thus, we will choose a point that is 1 unit from the origin. Finally, apply the definitions of the appropriate trigonometric functions.

EXAMPLE 2 Trigonometric Functions of Quadrantal Angles

Evaluate, if possible, the sine function and the tangent function at the following four quadrantal angles:

a. $\theta = 0° = 0$ **b.** $\theta = 90° = \frac{\pi}{2}$ **c.** $\theta = 180° = \pi$ **d.** $\theta = 270° = \frac{3\pi}{2}$.

Solution

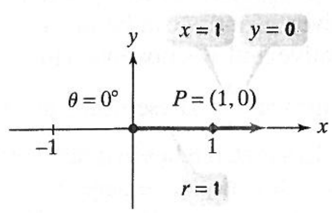


Figure 5.34

a. If $\theta = 0° = 0$ radians, then the terminal side of the angle is on the positive x-axis. Let us select the point $P = (1, 0)$ with $x = 1$ and $y = 0$. This point is 1 unit from the origin, so $r = 1$. **Figure 5.34** shows values of x, y, and r corresponding to $\theta = 0°$ or 0 radians. Now that we know x, y, and r, we can apply the definitions of the sine and tangent functions.

$$\sin 0° = \sin 0 = \frac{y}{r} = \frac{0}{1} = 0$$

$$\tan 0° = \tan 0 = \frac{y}{x} = \frac{0}{1} = 0$$

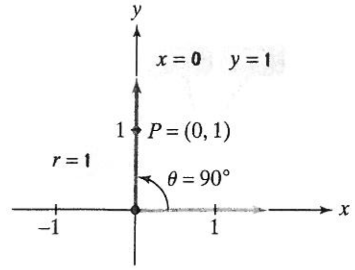


Figure 5.35

b. If $\theta = 90° = \frac{\pi}{2}$ radians, then the terminal side of the angle is on the positive y-axis. Let us select the point $P = (0, 1)$ with $x = 0$ and $y = 1$. This point is 1 unit from the origin, so $r = 1$. **Figure 5.35** shows values of x, y, and r corresponding to $\theta = 90°$ or $\frac{\pi}{2}$. Now that we know x, y, and r, we can apply the definitions of the sine and tangent functions.

$$\sin 90° = \sin\frac{\pi}{2} = \frac{y}{r} = \frac{1}{1} = 1$$

$$\tan 90° = \tan\frac{\pi}{2} = \frac{y}{x} = \cancel{\frac{1}{0}}$$

Because division by 0 is undefined, tan 90° is undefined.

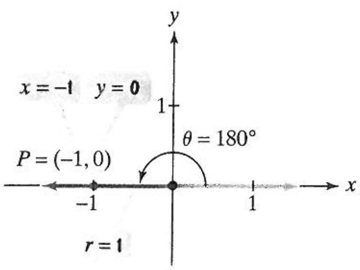


Figure 5.36

c. If $\theta = 180° = \pi$ radians, then the terminal side of the angle is on the negative x-axis. Let us select the point $P = (-1, 0)$ with $x = -1$ and $y = 0$. This point is 1 unit from the origin, so $r = 1$. **Figure 5.36** shows values of x, y, and r corresponding to $\theta = 180°$ or π. Now that we know x, y, and r, we can apply the definitions of the sine and tangent functions.

$$\sin 180° = \sin\pi = \frac{y}{r} = \frac{0}{1} = 0$$

$$\tan 180° = \tan\pi = \frac{y}{x} = \frac{0}{-1} = 0$$

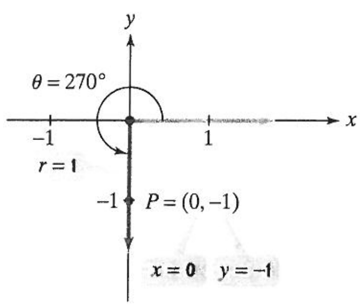


Figure 5.37

d. If $\theta = 270° = \frac{3\pi}{2}$ radians, then the terminal side of the angle is on the negative y-axis. Let us select the point $P = (0, -1)$ with $x = 0$ and $y = -1$. This point is 1 unit from the origin, so $r = 1$. **Figure 5.37** shows values of x, y, and r corresponding to $\theta = 270°$ or $\frac{3\pi}{2}$. Now that we know x, y, and r, we can apply the definitions of the sine and tangent functions.

$$\sin 270° = \sin\frac{3\pi}{2} = \frac{y}{r} = \frac{-1}{1} = -1$$

$$\tan 270° = \tan\frac{3\pi}{2} = \frac{y}{x} = \cancel{\frac{-1}{0}}$$

Because division by 0 is undefined, tan 270° is undefined.

Discovery

Try finding tan 90° and tan 270° with your calculator. Describe what occurs.

Check Point 2 Evaluate, if possible, the cosine function and the cosecant function at the following four quadrantal angles:

a. $\theta = 0° = 0$ **b.** $\theta = 90° = \frac{\pi}{2}$ **c.** $\theta = 180° = \pi$ **d.** $\theta = 270° = \frac{3\pi}{2}$.

2 Use the signs of the trigonometric functions.

The Signs of the Trigonometric Functions

In Example 2, we evaluated trigonometric functions of quadrantal angles. However, we will now return to the trigonometric functions of nonquadrantal angles. **If θ is not a quadrantal angle, the sign of a trigonometric function depends on the quadrant in which θ lies.** In all four quadrants, r is positive. However, x and y can be positive or negative. For example, if θ lies in quadrant II, x is negative and y is positive. Thus, the only positive ratios in this quadrant are $\frac{y}{r}$ and its reciprocal, $\frac{r}{y}$. These ratios are the function values for the sine and cosecant, respectively. In short, if θ lies in quadrant II, $\sin\theta$ and $\csc\theta$ are positive. The other four trigonometric functions are negative.

Figure 5.38 summarizes the signs of the trigonometric functions. If θ lies in quadrant I, all six functions are positive. If θ lies in quadrant II, only $\sin\theta$ and $\csc\theta$ are positive. If θ lies in quadrant III, only $\tan\theta$ and $\cot\theta$ are positive. Finally, if θ lies in quadrant IV, only $\cos\theta$ and $\sec\theta$ are positive. Observe that the positive functions in each quadrant occur in reciprocal pairs.

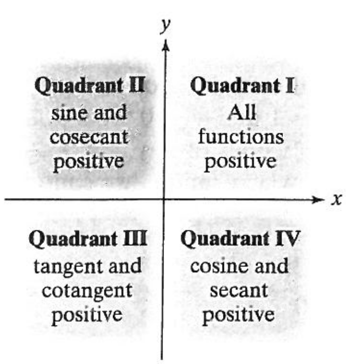


Figure 5.38 The signs of the trigonometric functions

Study Tip

Here's a phrase to help you remember the signs of the trig functions:

A	Smart	Trig	Class.
All trig functions are positive in QI.	Sine and its reciprocal, cosecant, are positive in QII.	Tangent and its reciprocal, cotangent, are positive in QIII.	Cosine and its reciprocal, secant, are positive in QIV.

EXAMPLE 3 Finding the Quadrant in Which an Angle Lies

If $\tan\theta < 0$ and $\cos\theta > 0$, name the quadrant in which angle θ lies.

Solution When $\tan\theta < 0$, θ lies in quadrant II or IV. When $\cos\theta > 0$, θ lies in quadrant I or IV. When both conditions are met ($\tan\theta < 0$ and $\cos\theta > 0$), θ must lie in quadrant IV.

Check Point 3 If $\sin\theta < 0$ and $\cos\theta < 0$, name the quadrant in which angle θ lies.

EXAMPLE 4 Evaluating Trigonometric Functions

Given $\tan\theta = -\frac{2}{3}$ and $\cos\theta > 0$, find $\cos\theta$ and $\csc\theta$.

Solution Because the tangent is negative and the cosine is positive, θ lies in quadrant IV. This will help us to determine whether the negative sign in $\tan\theta = -\frac{2}{3}$ should be associated with the numerator or the denominator. Keep in mind that in quadrant IV, x is positive and y is negative. Thus,

In quadrant IV, y is negative.

$$\tan\theta = -\frac{2}{3} = \frac{y}{x} = \frac{-2}{3}.$$

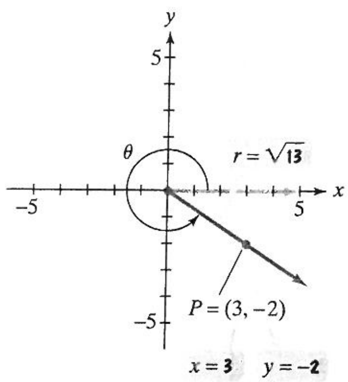


Figure 5.39 $\tan\theta = -\frac{2}{3}$ and $\cos\theta > 0$

(See **Figure 5.39**.) Thus, $x = 3$ and $y = -2$. Furthermore,

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13}.$$

Now that we know x, y, and r, we can find $\cos\theta$ and $\csc\theta$.

$$\cos\theta = \frac{x}{r} = \frac{3}{\sqrt{13}} = \frac{3}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}} = \frac{3\sqrt{13}}{13} \qquad \csc\theta = \frac{r}{y} = \frac{\sqrt{13}}{-2} = -\frac{\sqrt{13}}{2}$$

Check Point 4 Given $\tan\theta = -\frac{1}{3}$ and $\cos\theta < 0$, find $\sin\theta$ and $\sec\theta$.

In Example 4, we used the quadrant in which θ lies to determine whether a negative sign should be associated with the numerator or the denominator. Here's a situation, similar to Example 4, where negative signs should be associated with *both* the numerator and the denominator:

$$\tan\theta = \frac{3}{5} \quad \text{and} \quad \cos\theta < 0.$$

Because the tangent is positive and the cosine is negative, θ lies in quadrant III. In quadrant III, x is negative and y is negative. Thus,

$$\tan\theta = \frac{3}{5} = \frac{y}{x} = \frac{-3}{-5}.$$

We see that $x = -5$ and $y = -3$.

3 Find reference angles.

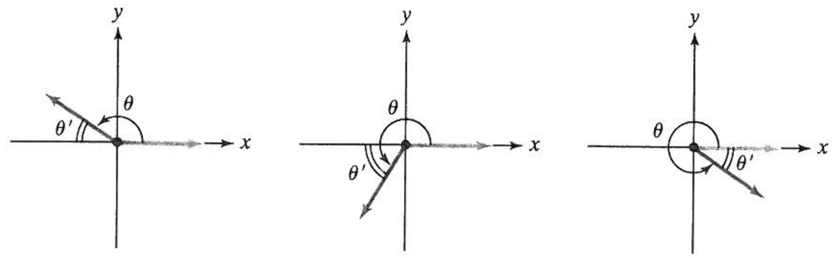
Reference Angles

We will often evaluate trigonometric functions of positive angles greater than 90° and all negative angles by making use of a positive acute angle. This positive acute angle is called a *reference angle.*

Definition of a Reference Angle

Let θ be a nonacute angle in standard position that lies in a quadrant. Its **reference angle** is the positive acute angle θ' formed by the terminal side of θ and the x-axis.

Figure 5.40 shows the reference angle for θ lying in quadrants II, III, and IV. Notice that the formula used to find θ', the reference angle, varies according to the quadrant in which θ lies. You may find it easier to find the reference angle for a given angle by making a figure that shows the angle in standard position. The acute angle formed by the terminal side of this angle and the x-axis is the reference angle.



If $90° < \theta < 180°$, then $\theta' = 180° - \theta$.

If $180° < \theta < 270°$, then $\theta' = \theta - 180°$.

If $270° < \theta < 360°$, then $\theta' = 360° - \theta$.

Figure 5.40 Reference angles, θ', for positive angles, θ, in quadrants II, III, and IV

EXAMPLE 5 Finding Reference Angles

Find the reference angle, θ', for each of the following angles:

a. $\theta = 345°$ **b.** $\theta = \frac{5\pi}{6}$ **c.** $\theta = -135°$ **d.** $\theta = 2.5$.

Solution

a. A 345° angle in standard position is shown in **Figure 5.41**. Because 345° lies in quadrant IV, the reference angle is

$$\theta' = 360° - 345° = 15°.$$

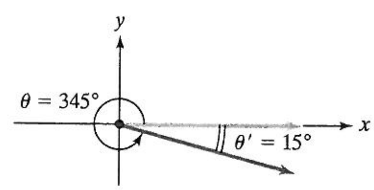


Figure 5.41

b. Because $\frac{5\pi}{6}$ lies between $\frac{\pi}{2} = \frac{3\pi}{6}$ and $\pi = \frac{6\pi}{6}$, $\theta = \frac{5\pi}{6}$ lies in quadrant II. The angle is shown in **Figure 5.42**. The reference angle is

$$\theta' = \pi - \frac{5\pi}{6} = \frac{6\pi}{6} - \frac{5\pi}{6} = \frac{\pi}{6}.$$

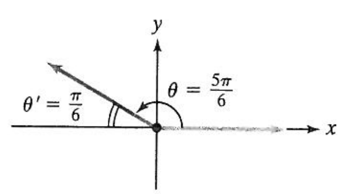


Figure 5.42

c. A $-135°$ angle in standard position is shown in **Figure 5.43**. The figure indicates that the positive acute angle formed by the terminal side of θ and the x-axis is 45°. The reference angle is

$$\theta' = 45°.$$

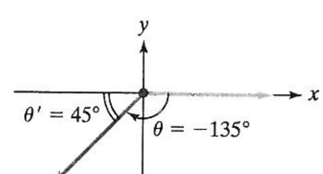


Figure 5.43

Discovery

Solve part (c) by first finding a positive coterminal angle for $-135°$ less than 360°. Use the positive coterminal angle to find the reference angle.